# Permutation Matrix

A *permutation matrix* is a square matrix[1] in which is zero everywhere apart from having only one '*1*' on every row and in every column. For example the following matrix is an example of a 3×3 permutation matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

When a matrix or vector is pre-multiplied[2] by a permutation matrix then its effect is to rearrange its rows. For example, pre-multiplying a typical matrix or vector by the example permutation matrix above is illustrated by the following examples:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 8 & 2 \\ 6 & 7 & 5 \\ 9 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 & 5 \\ 9 & 4 & 1 \\ 3 & 8 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 3 \end{pmatrix}.$$

In this example the effect of pre-multiplication with the example permutation matrix is as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \text{replace row 1 with row 2,} \\ \text{replace row 2 with row 3,} \\ \text{replace row 3 with row 1.} \end{matrix}$$

Hence a '1' in row $i$ and column $j$ in the permutation matrix has the effect of replacing row $i$ by row $j$ in the matrix or vector following pre-multiplication.

When a matrix or vector is post-multiplied by a permutation matrix then its effect is to rearrange its columns. For example, post-multiplying a typical matrix or vector by the example permutation matrix above is illustrated by the following examples:

$$\begin{pmatrix} 3 & 8 & 2 \\ 6 & 7 & 5 \\ 9 & 4 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 5 & 6 & 7 \\ 1 & 9 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 8 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \end{pmatrix}.$$

The effect of the post-multiplication of a matrix or vector by a permutation matrix with a '*1*' in row $i$ and column $j$ has the effect of replacing column $j$ by column $i$.

There are six 3×3 permutation matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence there are six possible outcomes of pre-multiplication with a 3×3 permutation matrix, there are six possible rearrangements of the rows, just as there are six possible permutations[3] of three items, hence the term *permutation matrix*. (The same point can be made for the effect on columns following the post-multiplication with the permutation matrices.) In general there are $n!$ $n \times n$ permutation matrices, just as there are $n!$ permutations or arrangements of $n$ objects.

[1] Matrix Definitions
[2] Matrix Arithmetic
[3] Permutations

**Transpose of a permutation matrix**

The transpose of a permutation matrix is also a permutation matrix. For the example permutation matrix above, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, the transpose is $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

The transpose of a permutation is also its inverse. For the example above:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In general if $P$ is a permutation matrix then $PP^T = P^TP = I$ and $P^T = P^{-1}$.